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STABILITY OF AN ACCELERATING BEAM

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Lyapunov's method has been used to derive a sufficient condition for the transverse stability of an axially accelerating beam. A multiple-time-scale formulation is also presented to study the stability when the beam has constant axial acceleration/deceleration. In such situations, an accelerating beam is found to be always stable, whereas a decelerating beam may undergo ephemeral instability. The non-linear terms do not affect the stability condition; they only change the frequency of oscillation.

1. INTRODUCTION

Most of the studies on the vibration of axially moving continuous systems (e.g. saw-bands, travelling threadlines, magnetic tapes, etc.) have been restricted to the beams and strings having uniform axial speed. Although the operating speed is normally maintained constant, the axial speed does vary considerably during the starting and stopping phases. For a travelling string, numerical studies have revealed that the response builds up or decays as the string is decelerated or accelerated respectively [1, 2]. In reference [2], an experiment with a pipe carrying fluid was also reported to validate the numerical results. Though the stability of a beam travelling with a periodically varying axial speed has been studied [3], that with a constant acceleration or deceleration has received little attention.

In the present work, the non-linear vibration of a travelling beam with constant axial acceleration or deceleration, maintaining the same direction of the axial movement, is considered. Lyapunov's method has been used to determine the sufficiency condition for asymptotic stability of a beam having any arbitrary acceleration. However, in some cases where the method cannot ascertain the stability, a perturbation method based on Multiple Time Scale (MTS) is used. The change in the speed due to constant acceleration or deceleration is treated as perturbation to the uniform speed.

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2. THEORETICAL ANALYSIS

2.1. EQUATION OF MOTION

The equation of motion of an accelerating beam, taking only the linear terms, is given by [4]

$$\rho A\left[\frac{\partial^2 w^*}{\partial t^2}+2c^*\frac{\partial^2 w^*}{\partial\xi\,\partial t}+c^{*2}\frac{\partial^2 w^*}{\partial\xi^2}\right]-T_0^*\frac{\partial^2 w^*}{\partial\xi^2}+EI_z\frac{\partial^4 w^*}{\partial\xi^4}+\rho A\frac{\mathrm{d}c^*}{\mathrm{d}t}\frac{\partial w^*}{\partial\xi}=0, \tag{1}$$

where the symbols are explained in the appendix. The boundary conditions are taken to be simply supported, i.e.

$$w^*(0,t)=w^*(l,t)=0$$

and

$$\frac{\partial^2 w^*(0,t)}{\partial\xi^2}=\frac{\partial^2 w^*(l,t)}{\partial\xi^2}=0.$$

Using the following non-dimensionalization scheme

$$\gamma=\frac{r}{l},\qquad w=\frac{w^*}{l\gamma^2},\qquad x=\frac{\xi}{l},\qquad \tau=\frac{1}{l}\left(\frac{E}{\rho}\right)^{1/2}\gamma t,$$

$$c=c^*\left(\frac{E}{\rho}\right)^{-1/2}\frac{1}{\gamma}\qquad\text{and}\qquad T_0=\frac{T_0^*}{EA\gamma^2},$$

equation (1) becomes

$$\frac{\partial^2 w}{\partial\tau^2}+2c\frac{\partial^2 w}{\partial x\partial\tau}+(c^2-T_0)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}+\frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial w}{\partial x}=0. \tag{2}$$

Under the usual assumptions [5], if the geometric non-linear term is included then the equation of motion is given by

$$\frac{\partial^2 w}{\partial\tau^2}+2c\frac{\partial^2 w}{\partial x\partial\tau}+(c^2-T_0)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}+\frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial w}{\partial x}=\varepsilon\left[\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2}, \tag{3}$$

where $\varepsilon(=\gamma^2/2)\ll 1$ is the small parameter. It is assumed that the transverse damping force on the beam is mainly governed by the particle velocity in the transverse direction. However, with a damping force assumed to be proportional to the total velocity [6, 7], the major conclusions of the subsequent analysis remain unchanged. Thus, introducing a small viscous damping term $\delta(\partial w/\partial\tau)$, the equation of motion turns out to be

$$\frac{\partial^2 w}{\partial\tau^2}+\delta\frac{\partial w}{\partial\tau}+2c\frac{\partial^2 w}{\partial x\partial\tau}+(c^2-T_0)\frac{\partial^2 w}{\partial x^2}+\frac{\partial^4 w}{\partial x^4}+\frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial w}{\partial x}=\varepsilon\left[\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]\frac{\partial^2 w}{\partial x^2} \tag{4}$$

which can be recast in the standard state-space form as

$$\mathbf{A}\frac{\partial \mathbf{W}}{\partial \tau}+\mathbf{B}\mathbf{W}+\mathbf{N}=0 \tag{5}$$

with

$$\mathbf{A}=\begin{bmatrix} I & 0\\ 0 & K\end{bmatrix},\qquad \mathbf{B}=\begin{bmatrix} G & K\\ -K & 0\end{bmatrix},\qquad \mathbf{W}=\begin{Bmatrix}\dfrac{\partial w}{\partial \tau}\\ w\end{Bmatrix}$$

where $K\equiv(c^2-T_0)\,\partial^2/\partial x^2+\partial^4/\partial x^4$, $G\equiv 2c(\partial/\partial x)$, I as the identity operator and

$$\mathbf{N}=\left\{\delta\frac{\partial w}{\partial \tau}+\frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial w}{\partial x}-\varepsilon\left(\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right)\frac{\partial^2 w}{\partial x^2},0\right\}^{\mathrm{T}}. \tag{6}$$

In the following, the stability of response $w(x,\tau)$ is analyzed using either equation (4) or (5), whichever is convenient depending on the method used.

2.2. STABILITY ANALYSIS USING LYAPUNOV'S METHOD

In this section, Lyapunov's direct method [8] is used to find the asymptotic nature of the response of the system governed by equation (4).

The Lyapunov function is taken as

$$V(w,\tau)=\frac{1}{2}\int_0^1\left(\frac{\partial w}{\partial \tau}\right)^2\mathrm{d}x+\frac{1}{2}(T_0-c^2)\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x$$
$$+\frac{1}{2}\int_0^1\left(\frac{\partial^2 w}{\partial x^2}\right)^2\mathrm{d}x+\frac{\varepsilon}{4}\left[\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right]^2. \tag{7}$$

Assuming that the axial speed always remains positive (i.e., $c>0$) and never crosses the first critical speed $\sqrt{T_0+\pi^2}$, the chosen Lyapunov function is positive definite, since $\int_0^1(\partial^2 w/\partial x^2)^2\,\mathrm{d}x\geqslant\pi^2\int_0^1(\partial w/\partial x)^2\,\mathrm{d}x$ for a continuous differentiable function with $w(0)=w(1)=0$ [9].

Equation (7) is differentiated along the curve satisfying the equation of motion. Subsequent integration with the help of non-dimensionalized boundary conditions yields

$$\frac{\mathrm{d}V}{\mathrm{d}\tau}=-\delta\int_0^1\left(\frac{\partial w}{\partial \tau}\right)^2\mathrm{d}x-2c\int_0^1\frac{\partial w}{\partial \tau}\frac{\partial^2 w}{\partial x\,\partial \tau}\mathrm{d}x$$
$$-c\frac{\mathrm{d}c}{\mathrm{d}\tau}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{\mathrm{d}c}{\mathrm{d}\tau}\int_0^1\frac{\partial w}{\partial x}\frac{\partial w}{\partial \tau}\mathrm{d}x.$$

It can be seen by simple integration by parts that the term $\int_0^1(\partial w/\partial \tau)(\partial^2 w/\partial x\,\partial \tau)\,\mathrm{d}x=0$ for the given boundary conditions. Thus,

$$\frac{\mathrm{d}V}{\mathrm{d}\tau}=-\delta\int_0^1\left(\frac{\partial w}{\partial \tau}\right)^2\mathrm{d}x-c\frac{\mathrm{d}c}{\mathrm{d}\tau}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x-\frac{\mathrm{d}c}{\mathrm{d}\tau}\int_0^1\frac{\partial w}{\partial x}\frac{\partial w}{\partial \tau}\mathrm{d}x. \tag{8}$$

Rearranging equation (8) as

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} = -\left[\int_0^1 \left(\sqrt{\delta}\,\frac{\partial w}{\partial \tau} + \frac{1}{2\sqrt{\delta}}\frac{\mathrm{d}c}{\mathrm{d}\tau}\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x + \left\{c\,\frac{\mathrm{d}c}{\mathrm{d}\tau} - \frac{1}{4\delta}\left(\frac{\mathrm{d}c}{\mathrm{d}\tau}\right)^2\right\}\int_0^1 \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\right] \tag{9}$$

it can be seen that for a continuously accelerating beam, i.e., $\mathrm{d}c/\mathrm{d}\tau > 0$,

$$\frac{\mathrm{d}V}{\mathrm{d}\tau} < 0 \quad \text{if} \left(\frac{\mathrm{d}c}{\mathrm{d}\tau}\right)^2 < 4\delta c\left(\frac{\mathrm{d}c}{\mathrm{d}\tau}\right). \tag{10}$$

When $\mathrm{d}c/\mathrm{d}\tau = 0$, it is observed from equation (9) that $\mathrm{d}V/\mathrm{d}\tau < 0$, i.e., the system is asymptotically stable. When $\mathrm{d}c/\mathrm{d}\tau \neq 0$, the system is stable if

$$\left(\frac{\mathrm{d}c}{\mathrm{d}\tau}\right) < 4\delta c. \tag{11}$$

Further, it is seen that the beam is stable (i.e., $\mathrm{d}V/\mathrm{d}\tau < 0$) when both $c < 0$ and $\mathrm{d}c/\mathrm{d}\tau < 0$ and the inequality (10) is satisfied. The situation corresponds to an accelerating beam moving in the direction of $-x$. Thus, considering the symmetric nature of the boundary conditions, these two cases are physically identical.

Although the stability is confirmed if condition (11) is satisfied, nothing can be said if the same condition is violated. Similarly, for a decelerating beam (i.e., $\mathrm{d}c/\mathrm{d}\tau < 0$ but $c > 0$), equation (9) cannot be used to judge the asymptotic stability. This necessitates the choice of other functions. For such situations, however, instead of searching for different Lyapunov functions, a perturbation analysis to determine the stability is presented in the following section.

2.3. STABILITY ANALYSIS USING THE MTS METHOD

In this section the stability of a uniformly accelerating or decelerating beam is analyzed using the multiple-time-scale method. The change in speed is treated as a perturbation to the uniform speed. Furthermore, the damping is assumed to be small. Consequently, the speed of the beam and the damping factor are written, respectively, as

$$c = c_0 + \varepsilon\alpha\tau, \tag{12}$$

and

$$\delta = \varepsilon\delta_0, \tag{13}$$

where $|\alpha|$ governs the magnitude of constant acceleration/deceleration. Using two time scales $t_0 = \tau$ and $t_1 = \varepsilon\tau$, the partial differentials in time (τ) can be written up to $o(\varepsilon)$, as

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t_0} + \varepsilon\,\frac{\partial}{\partial t_1}, \tag{14}$$

$$\frac{\partial^2}{\partial \tau^2} = \frac{\partial^2}{\partial t_0^2} + 2\varepsilon\,\frac{\partial^2}{\partial t_0\,\partial t_1} \tag{15}$$

and the displacement as

$$w(x, t_0, t_1) = w_0(x, t_0, t_1) + \varepsilon w_1(x, t_0, t_1). \tag{16}$$

By substituting equations (12)–(16) into equation (4) and equating the coefficients of the like powers of ε from both sides, the following equations (in the state-space form) are obtained:

$$\varepsilon^0 : \mathbf{A} \frac{\partial \mathbf{W}_0}{\partial t_0} + \mathbf{B}\mathbf{W}_0 = 0, \tag{17}$$

$$\varepsilon^1 : \mathbf{A} \frac{\partial \mathbf{W}_1}{\partial t_0} + \mathbf{B}\mathbf{W}_1 + \mathbf{N}_0 = 0, \tag{18}$$

where

$$\mathbf{N}_0 = \left\{ \delta_0 \frac{\partial w_0}{\partial t_0} + 2\alpha t_0 \frac{\partial^2 w_0}{\partial x \partial t_0} + 2c_0 \frac{\partial^2 w_0}{\partial x \partial t_1} + 2 \frac{\partial^2 w_0}{\partial t_0 \partial t_1} + 2\alpha c_0 t_0 \frac{\partial^2 w_0}{\partial x^2} + \alpha \frac{\partial w_0}{\partial x} \right.$$
$$\left. - \left(\int_0^1 \left(\frac{\partial w_0}{\partial x} \right)^2 \mathrm{d}x \right) \frac{\partial^2 w_0}{\partial x^2}, 0 \right\}^{\mathrm{T}}. \tag{19}$$

Equation (17) is solved to yield

$$\mathbf{W}_0(x, t_0, t_1) = \sum_{n=1}^{\infty} \left[\frac{a_n(t_1)}{2} \Phi_n(x) \, \mathrm{e}^{\mathrm{i}\omega_n^l t_0} + \frac{\bar{a}_n(t_1)}{2} \bar{\Phi}_n(x) \, \mathrm{e}^{-\mathrm{i}\omega_n^l t_0} \right], \tag{20}$$

where $a_n(t_1)$ is the complex amplitude and the bar at the top denotes the complex conjugate. In equation (20)

$$\Phi_n \left(= \begin{Bmatrix} \mathrm{i}\omega_n^l \phi_n \\ \phi_n \end{Bmatrix} \right)$$

is the modal vector with ϕ_n as the complex mode shape and ω_n^l as the corresponding linear natural frequency. Further, the following orthogonality relations are well-known [5]:

$$\int_0^1 \bar{\Phi}_n^{\mathrm{T}} \mathbf{A} \Phi_m \, \mathrm{d}x = 0 \quad \text{for } n \neq m \tag{21}$$

and

$$\int_0^1 \Phi_n^{\mathrm{T}} \mathbf{A} \Phi_m \, \mathrm{d}x = 0 \quad \text{for all } n \text{ and } m. \tag{22}$$

To solve equation (18), first equation (20) is substituted into equation (19) and $\mathbf{N}_0$ is expressed as

$$\mathbf{N}_0 = \sum_{n=1}^{\infty} \mathbf{P}_n(x) \mathrm{e}^{\mathrm{i}\omega_n^l t_0} + \mathbf{Q}(x, t_0) + \text{c.c.},$$

where c.c. denotes the complex conjugate of the previous terms and $\mathbf{Q}$ contains the terms having all the frequencies except ω_n^l. One readily finds $\mathbf{P}_n$ as

$$\mathbf{P}_n(x) = \left\{ \mathrm{i}\omega_n^l \delta_0 \frac{a_n}{2}\phi_n + \mathrm{i}\pi\alpha a_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x} + c_0 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\frac{\mathrm{d}a_n}{\mathrm{d}t_1} \right.$$

$$+ \mathrm{i}\omega_n^l \phi_n \frac{\mathrm{d}a_n}{\mathrm{d}t_1} + c_0 \alpha a_n \frac{\pi}{\omega_n^l}\frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2} + \alpha\frac{a_n}{2}\frac{\mathrm{d}\phi_n}{\mathrm{d}x}$$

$$\left. + \frac{a_n^2 \bar{a}_n}{4}\left[\left(2\int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\frac{\mathrm{d}\bar{\phi}_n}{\mathrm{d}x}\mathrm{d}x\right)\frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2} + \left(\int_0^1 \left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)\frac{\mathrm{d}^2\bar{\phi}_n}{\mathrm{d}x^2}\right], 0\right\}^{\mathrm{T}}. \quad (23)$$

Using the orthogonality relations, $\mathbf{P}_n$ is decomposed in terms of the linear modes as

$$\mathbf{P}_n(x) = \sum_{m=1}^{\infty} [p_{mn}\Phi_m(x) + q_{mn}\bar{\Phi}_n(x)],$$

where

$$p_{mn} = \frac{\int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{A}\mathbf{P}_n \,\mathrm{d}x}{\int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{A}\Phi_m \,\mathrm{d}x}$$

and

$$q_{mn} = \frac{\int_0^1 \Phi_m^{\mathrm{T}} \mathbf{A}\mathbf{P}_n \,\mathrm{d}x}{\int_0^1 \bar{\Phi}_m^{\mathrm{T}} \mathbf{A}\Phi_m \,\mathrm{d}x}.$$

Now, secular terms will be avoided in the solution of equation (18) if

$$p_{nn} = 0,$$

i.e.,

$$\int_0^1 \bar{\Phi}_n^{\mathrm{T}} \mathbf{A}\mathbf{P}_n \,\mathrm{d}x = 0. \quad (24)$$

Equation (24) can be expanded with the help of equation (23) as

$$\frac{1}{2}\left(2\omega_n^l \int_0^1 \bar{\phi}_n\phi_n \,\mathrm{d}x - 2\mathrm{i}c_0 \int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\bar{\phi}_n \,\mathrm{d}x\right)\frac{\mathrm{d}a_n}{\mathrm{d}t_1}$$

$$= -\omega_n^l \frac{a_n}{2}\delta_0 \int_0^1 \bar{\phi}_n\phi_n \,\mathrm{d}x - \pi\alpha a_n \int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\bar{\phi}_n \,\mathrm{d}x + \mathrm{i}c_0\alpha a_n \frac{\pi}{\omega_n^l}\int_0^1 \frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2}\bar{\phi}_n \,\mathrm{d}x$$

$$+ \mathrm{i}\alpha\frac{a_n}{2}\int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\bar{\phi}_n \,\mathrm{d}x - \frac{\mathrm{i}}{8}a_n^2\bar{a}_n\left[\left(2\int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\frac{\mathrm{d}\bar{\phi}_n}{\mathrm{d}x}\mathrm{d}x\right)\left(\int_0^1 \bar{\phi}_n \frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2}\mathrm{d}x\right)\right.$$

$$\left. + \left(\int_0^1 \left(\frac{\mathrm{d}\phi_n}{\mathrm{d}x}\right)^2 \mathrm{d}x\right)\left(\int_0^1 \bar{\phi}_n \frac{\mathrm{d}^2\bar{\phi}_n}{\mathrm{d}x^2}\mathrm{d}x\right)\right]. \quad (25)$$

It is to be noted that for a travelling beam,

$$\mathrm{Re}\left[\int_0^1 \bar{\phi}_n \frac{\mathrm{d}\phi_n}{\mathrm{d}x}\mathrm{d}x\right] = 0, \qquad \mathrm{Im}\left[\int_0^1 \bar{\phi}_n\phi_n \,\mathrm{d}x\right] = 0 \quad \text{and} \quad \mathrm{Im}\left[\int_0^1 \bar{\phi}_n \frac{\mathrm{d}^2\phi_n}{\mathrm{d}x^2}\mathrm{d}x\right] = 0.$$

Now by substituting $a_n = \bar{a}_n e^{i\theta_n}$ into equation (25) and equating, separately, the real and imaginary parts from both sides, the following relations are obtained:

$$\frac{1}{2}\left(2\omega_n^l \int_0^1 \bar{\phi}_n \phi_n \, dx - 2ic_0 \int_0^1 \frac{d\phi_n}{dx} \bar{\phi}_n \, dx\right) \frac{d\bar{a}_n}{dt_1} = -\omega_n^l \frac{\tilde{a}_n}{2} \delta_0 \int_0^1 \bar{\phi}_n \phi_n \, dx + i\alpha \frac{\tilde{a}_n}{2} \int_0^1 \frac{d\phi_n}{dx} \bar{\phi}_n \, dx \quad (26)$$

and

$$\frac{1}{2}\left(2\omega_n^l \int_0^1 \bar{\phi}_n \phi_n \, dx - 2ic_0 \int_0^1 \frac{d\phi_n}{dx} \bar{\phi}_n \, dx\right) \frac{d\theta_n}{dt_1} \tilde{a}_n = i\pi\alpha\tilde{a}_n \int_0^1 \frac{d\phi_n}{dx} \bar{\phi}_n \, dx + c_0\alpha\tilde{a}_n \frac{\pi}{\omega_n^l} \int_0^1 \frac{d^2\phi_n}{dx^2} \bar{\phi}_n \, dx - \frac{1}{8}\tilde{a}_n^3 \left[\left(2\int_0^1 \frac{d\phi_n}{dx}\frac{d\bar{\phi}_n}{dx} \, dx\right)\left(\int_0^1 \bar{\phi}_n \frac{d^2\phi_n}{dx^2} \, dx\right) + \left(\int_0^1 \left(\frac{d\phi_n}{dx}\right)^2 dx\right)\left(\int_0^1 \bar{\phi}_n \frac{d^2\bar{\phi}_n}{dx^2} \, dx\right)\right]. \quad (27)$$

Writing $\int_0^1 \bar{\phi}_n (d\phi_n/dx) \, dx = iS$ $(S > 0)$, equation (26) can be recast in the following form:

$$\frac{1}{2}\left(2\omega_n^l \int_0^1 \bar{\phi}_n \phi_n \, dx - 2ic_0 \int_0^1 \frac{d\phi_n}{dx} \bar{\phi}_n \, dx\right) \frac{d\tilde{a}_n}{dt_1} = -\omega_n^l \frac{\tilde{a}_n}{2} \delta_0 \int_0^1 \bar{\phi}_n \phi_n \, dx - \alpha \frac{\tilde{a}_n}{2} S. \quad (28)$$

Obvious from equation (28) is the fact that for $\alpha > 0$, i.e., during acceleration, $d\tilde{a}_n/dt_1 < 0$ or the response decays. But for a decelerating beam (i.e., $\alpha < 0$), the response may be stable or unstable accordingly as

$$\omega_n^l \delta_0 \int_0^1 \bar{\phi}_n \phi_n \, dx > |\alpha| S \quad (29)$$

or

$$\omega_n^l \delta_0 \int_0^1 \bar{\phi}_n \phi_n \, dx < |\alpha| S, \quad (30)$$

respectively. However, the instability lasts only a short time. The ephemerally observed characteristics of the instability can be explained as follows. As the beam becomes unstable, the amplitude grows but the speed (remaining positive) also decreases since the beam decelerates. The value of S, obtained numerically, is found to decrease monotonically with decreasing speed. Consequently, with decreasing speed, the beam eventually regains its stability as $d\tilde{a}_n/dt_1$ becomes negative. Thus, the amplitude builds up to a limiting value before it starts decreasing. This phenomenon was observed in a pipe carrying fluid when the flow was stopped [2].

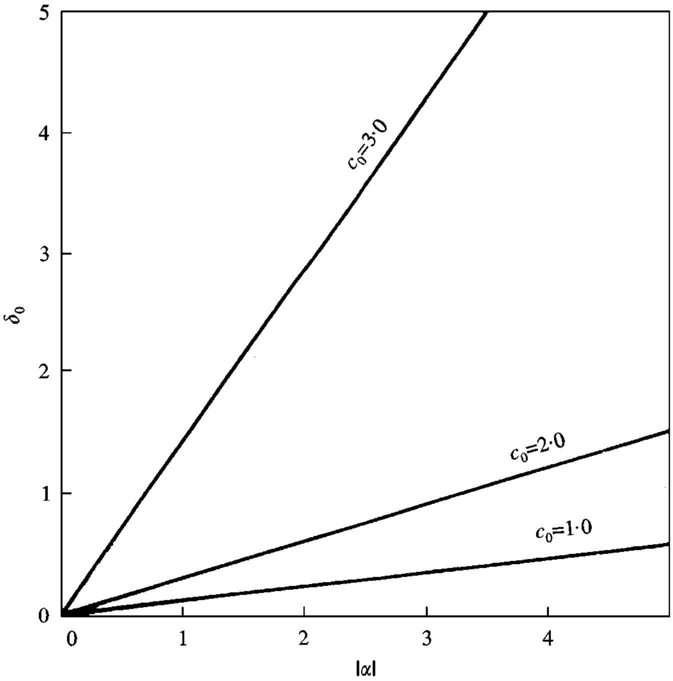

Figure 1. Stability boundaries for a uniformly decelerating beam.

It should be pointed out that the non-linear terms do not affect the stability. They, as seen from equation (27), only change the frequency of oscillation.

2.4. NUMERICAL RESULTS AND DISCUSSION

Numerical results are presented in this section for a beam having an initial tension $T_0 = 1$. For a beam with its axial speed given by equation (12), the asymptotic stability is confirmed using Lyapunov's method (see equation (9)) if

$$\delta_0 > \frac{\alpha}{4c_0}.$$

The MTS method, on the other hand, predicts the stability for all possible values of δ_0 ($\delta_0 > 0$). For a decelerating beam, the chosen Lyapunov function (see equation (7)) fails to ascertain the stability. However, it is evident from the MTS method (see equations (29) and (30)) that the stability depends on the values of δ_0 and $|\alpha|$. Furthermore, it is obvious from equations (29) and (30) that in the parameter space $(\delta_0, |\alpha|)$ the boundary delineating the stable and unstable regions is a straight line. Figure 1 shows these boundaries for three different values of c_0. The region above the line (representing the boundary) is stable. As expected, the damping required to prevent instability increases with increasing initial speed c_0.

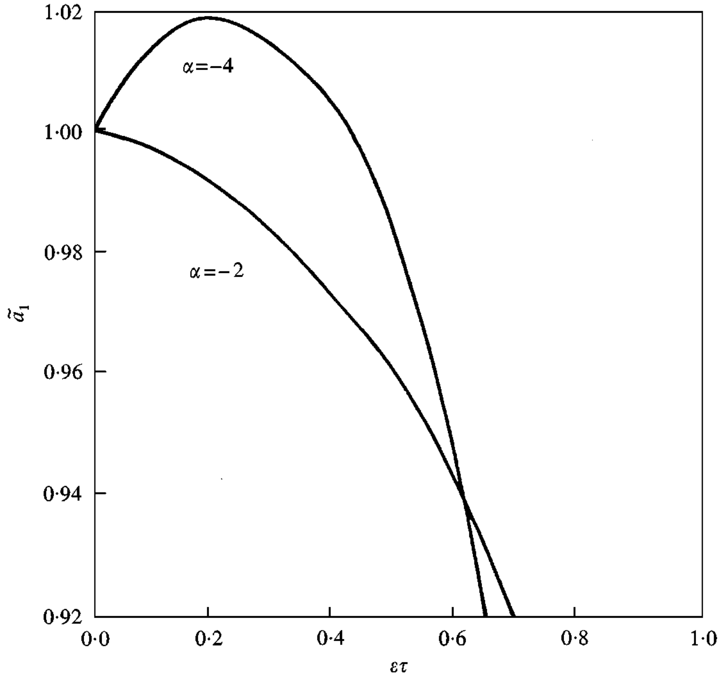

Figure 2. Variation of the response amplitude for a uniformly decelerating beam. $\delta_0 = 0{\cdot}7$.

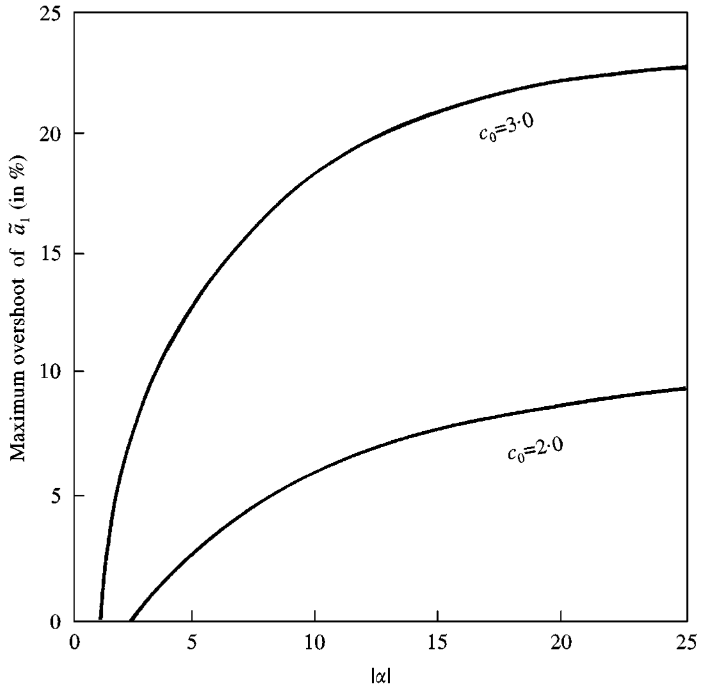

Figure 3. Variation of the maximum overshoot of response amplitude with the magnitude of uniform deceleration. $\delta_0 = 0{\cdot}7$.

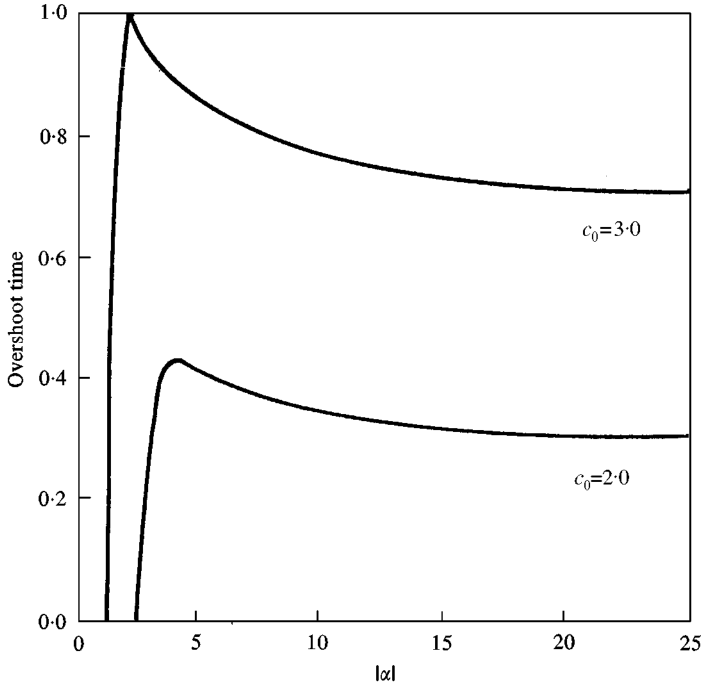

Figure 4. Variation of the overshoot time with the magnitude of uniform deceleration. $\delta_0 = 0{\cdot}7$.

The asymptotic or long-term stability, discussed in the previous section, is shown in Figure 2. The amplitude is approximately calculated as

$$\tilde{a}_n = \tilde{a}_n(0)\mathrm{e}^{-\int_0^\tau \Delta(\tau_1)\,\mathrm{d}\tau_1},$$

where

$$\Delta = \left(\omega_n^l \delta_0 \int_0^1 \bar{\phi}_n \phi_n \,\mathrm{d}x + \alpha S\right) \Bigg/ \left(2\omega_n^l \int_0^1 \bar{\phi}_n \phi_n \,\mathrm{d}x - 2\mathrm{i}c_0 \int_0^1 \frac{\mathrm{d}\phi_n}{\mathrm{d}x} \bar{\phi}_n \,\mathrm{d}x\right).$$

When the damping is not sufficiently large, the response amplitude shows a temporal rise. The results are obtained by assuming that only the first mode is excited. For the values of damping and axial speed considered, other modes are found to be stable.

The effects of deceleration magnitude on the maximum amplitude rise and the overshoot-time (i.e., the interval during which the response amplitude remains higher than the initial value) have been plotted in Figures 3 and 4 respectively.

3. CONCLUSION

The stability of the transverse vibration of a beam having non-uniform axial speed has been studied using Lyapunov's method. A sufficient condition for the stability of an accelerating beam has been obtained. Although this technique does not presuppose any special form of acceleration, a single Lyapunov function cannot ascertain the stability for all the values of acceleration or deceleration. A multiple-time-scale method has been used to adjudge the stability of a beam having constant acceleration/deceleration. Such an accelerating beam remains stable, but for a decelerating beam, instability may appear depending upon the values of the damping and the deceleration. The non-linear term does not play any role so far as the stability is concerned. It merely changes the frequency of oscillation. However, the amplitude of a decelerating beam does not grow unboundedly and the beam regains its stability in the long run.

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APPENDIX: NOMENCLATURE

w^* transverse displacement of beam
c^* uniform axial speed
T_0^* initial tension in the beam
ρ density of beam material
E Young's modulus of beam material
A area of the cross-section of the beam
l length of the beam
I_z second moment of area of cross-section about the neutral axis
r radius of gyration of beam cross-section $= \sqrt{I_z/A}$
γ slenderness ratio, $r/l \ll 1$
ε $= \gamma^2/2$
ξ longitudinal distance of a point on the beam from left support
t time
x non-dimensional distance
τ non-dimensional time
w non-dimensional transverse displacement

c	non-dimensional axial speed
T_0	non-dimensional tension
ϕ_n	nth linear complex normal mode $= \phi_n^R + \mathrm{i}\phi_n^I$
i	$= \sqrt{-1}$
ω_n^l	linear natural frequency of nth linear mode